Laplacian spectral characterization of some graph products

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Abstract

This paper studies the Laplacian spectral characterization of some graph products. We consider a class of connected graphs: $\mathscr{G} = \{G : |EG| \le |VG| + 1\}$, and characterize all graphs $G \in \mathscr{G}$ such that the products $G \times K_m$ are L-DS graphs. The main result of this paper states that, if $G \in \mathscr{G}$, except for C_6 and $\Theta_{3,2,5}$, is L-DS graph, so is the product $G \times K_m$. In addition, the L-cospectral graphs with $C_6 \times K_m$ and $\Theta_{3,2,5} \times K_m$ have been found.

Keywords: Laplacian Spectrum; L-cospectral graphs; L-DS graph

AMS classification: 05C50

1 Introduction

We start with some basic conceptions of graphs followed from [1]. Let $G = (VG, EG)$ be a graph with vertex set VG and edge set EG, where EG is a collection of 2-subsets of VG. All graphs considered here are simple and undirected. The *adjacency matrix* $A(G) = (a_{u,v})$ $(u, v \in VG)$ of G is a matrix whose rows and columns are labeled by VG, with $a_{u,v} = 1$ if $\{u, v\} \in EG$ and $a_{u,v} = 0$ otherwise. The matrix $L(G) = D(G) - A(G)$ is called the *Laplacian matrix* of G, where $D(G)$ is a diagonal matrix whose diagonal entry is the degree of the corresponding vertex. Since the matrix $L(G)$ is real and symmetric, its eigenvalues are real numbers and called the *Laplacian eigenvalues* of G. It can be shown that $L(G)$ is positive semidefinite. Assuming that $\lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_n (= 0)$ are these eigenvalues, the multiset $\mathrm{Spec}(G) = \{\lambda_1, \ldots, \lambda_n\}$ is called the *Laplacian spectrum* of G. For simplicity, we write $[\lambda_i]^{m_i} \in \mathrm{Spec}(G)$ to denote that the multiplicity of λ_i is m_i. Two graphs are said to be *L-cospectral* if they share the same Laplacian spectrum. Two graphs G and H are said to be *isomorphic* if there is a bijection between VG and VH which induces a bijection between EG and EH. Throughout this paper, we write $G = H$ whenever G and H are isomorphic. A graph G is called to be *determined by its Laplacian spectrum*, or *L-DS graph* for short, if all graphs L-cospectral with G are isomorphic to G.

Given two graphs G_1 and G_2 with disjoint vertex sets VG_1 and VG_2 and edge sets EG_1 and EG_2, the *disjoint union*, or *addition* for convenience, of G_1 and G_2 is defined to be the graph $G = (VG_1 \cup VG_2, EG_1 \cup EG_2)$, denoted by $G_1 + G_2$. Especially, $\underbrace{G + \cdots + G}_{m}$ is denoted by mG. The *product* of graphs G_1 and G_2 is the graph $G_1 + G_2$ together with all the edges joining VG_1 and VG_2, denoted by $G_1 \times G_2$. Let K_m be the *complete graph* of m vertices, P_m the *path* of m vertices, and C_m the *cycle* of m vertices, respectively. Clearly, the complete graph K_m can be written as the product of m isolated vertices. Let K_1 be an isolated vertex, then $K_m = \underbrace{K_1 \times \cdots \times K_1}_{m}$. Similarly, $mK_1 = \underbrace{K_1 + \cdots + K_1}_{m}$ denotes the disjoint union of m isolated vertices. A connected graph is called a *tree* if it contains no cycle, *unicyclic* if exactly one cycle, and *bicyclic* if two independent cycles. Let G be a connected graph. A subgraph S of G is called a *spanning tree* of G if

S is a tree and $VS = VG$. Denote by $s(G)$ the number of spanning trees of G. Obviously, $s(G) = 0$ if G is disconnected. These notations will be fixed throughout this paper.

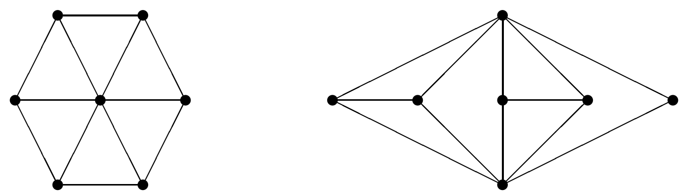
This paper is to characterize which graph products are determined by their Laplacian spectra. It is motivated by [5, 15] that we propose the following problem.

Problem 1. *Characterize all graphs G such that $G \times K_m$ are L-DS graphs.*

In [15], the wheel graph $C_n \times K_1$ for $n \neq 6$ is proved to be L-DS graph. In the conclusion, the authors posed an interesting question. The question is that which graphs satisfy the following relation:

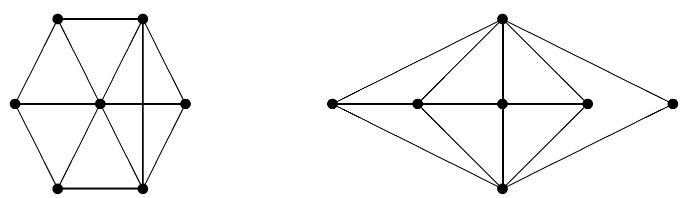
Relation 1. *If G is a L-DS graph, then $G \times K_1$ is also a L-DS graph.*

Clearly, Relation 1 is just a special case of Problem 1. It is known that if G is disconnected, i.e., G has at least two components, then G always satisfies Relation 1 (see Proposition 4 in [13]). If G is connected, we know that cycle C_n with $n \neq 6$ and path P_n satisfy Relation 1 [5, 15].



$C_6 \times K_1$ $\qquad$ $H_1 = 2K_1 \times (2P_2 + K_1)$

Figure 1: The L-cospectral graphs $C_6 \times K_1$ and H_1



$\Theta_{3,2,5} \times K_1$ $\qquad$ $H_2 = 2K_1 \times (P_4 + K_1)$

Figure 2: The L-cospectral graphs $\Theta_{3,2,5} \times K_1$ and H_2

In this paper, we consider a class of connected graphs: $\mathscr{G} = \{G : |EG| \leq |VG| + 1\}$, and characterize all graphs G among $\mathscr{G}$ such that $G \times K_m$ are L-DS graphs. Indeed, $\mathscr{G}$ consists of all connected trees, connected unicyclic graphs and connected bicyclic graphs. To characterize which connected trees satisfy Problem 1 are investigated in Section 3. And we show that if a connected tree T is L-DS, so is $T \times K_m$. The characterization for unicyclic graphs are investigated in Section 4. We prove that if a connected unicyclic graph $U \neq C_6$ is L-DS, then $U \times K_m$ is also L-DS. At last, we consider the products of L-DS bicyclic graphs and K_m. It is shown that all L-DS bicyclic graphs, except for $\Theta_{3,2,5}$, satisfy Problem 1, where $\Theta_{3,2,5}$ denotes the graph consisting of two cycles C_3 and C_5 who share a common path $P_2 = C_3 \cap C_5$. Meanwhile we find one new pair of L-cospectral graphs, which are $\Theta_{3,2,5} \times K_m$ and $H_2 \times K_{m-1}$, see Figure 2 for the case $m = 1$. Indeed, L-cospectral graphs shown in Figure 1, which are posed in [15], can also be figured out by our proof in Section 4.

2 Preliminaries

In this section, we mention some results, which will be used later.

Lemma 2.1. [1] *Let $\{\lambda_1, \ldots, \lambda_{n-1}, 0\}$ be the Laplacian spectrum of the graph G. Then*

$$s(G) = \frac{\lambda_1 \lambda_2 \cdots \lambda_{n-1}}{n}.$$

Lemma 2.2. [4,12] *Let G be a graph. The following can be determined by its Laplacian spectrum:*

(1) *The number of vertices of G.*
(2) *The number of edges of G.*
(3) *The number of components of G.*
(4) *The number of spanning trees of G.*
(5) *The sum of the squares of degrees of vertices.*

Lemma 2.3. [9] *Let G and H be two graphs with $|VG| = n$ and $|VH| = m$. Suppose* $\mathrm{Spec}(G) = \{\mu_1, \mu_2, \ldots, \mu_{n-1}, 0\}$ *and* $\mathrm{Spec}(H) = \{\nu_1, \nu_2, \ldots, \nu_{m-1}, 0\}$. *Then the Laplacian spectrum of the product $G \times H$ is*

$$\mathrm{Spec}(G \times H) = \{n+m, m+\mu_1, \ldots, m+\mu_{n-1}, n+\nu_1, \ldots, n+\nu_{m-1}, 0\}.$$

Lemma 2.4. *Suppose G is a L-DS graph. If there is a graph H and a positive integer m such that* $\mathrm{Spec}(G \times K_m) = \mathrm{Spec}(H \times K_m)$, *then we have $G = H$.*

Proof. Since $\mathrm{Spec}(G \times K_m) = \mathrm{Spec}(H \times K_m)$, Lemma 2.3 implies that $\mathrm{Spec}(G) = \mathrm{Spec}(H)$. Therefore, $G = H$ since G is a L-DS graph. □

Lemma 2.5. [2] *Let G be a connected graph with n vertices. Then n is the Laplacian eigenvalue with multiplicity k if and only if G is the product of exactly $k+1$ graphs.*

Lemma 2.6. [8] *Let G be a graph and $\lambda(G)$ the largest Laplacian eigenvalue of G. Denote by $d(v)$ the vertex degree of $v \in VG$. Then*

$$\lambda(G) \leq \max\{d(v) + m(v) | v \in VG\},$$

where $m(v) = \dfrac{1}{d(v)} \sum_{\{u,v\} \in EG} d(u)$ is the average of degrees for all neighbors of v.

Lemma 2.7. [7] *Let $\lambda(G)$ and $\Delta(G)$ be the maximum Laplacian eigenvalue and the maximum vertex degree of G, respectively. If G has at least one edge, then $\lambda(G) \geq \Delta(G) + 1$. Moreover, if G is connected graph of n vertices with $n > 1$, then we have*

$$\lambda(G) = \Delta(G) + 1 \iff \Delta(G) = n - 1.$$

3 Laplacian spectral characterization of the products of trees and complete graphs

In this section, the main result states that the products of L-DS trees and complete graphs are L-DS graphs. To prove this result, we first need one number theoretic proposition.

Proposition 3.1. *Let s and t be two positive integers. If $x_0, x_1, \ldots, x_k$ is a sequence of integers with $\sum_{i=0}^{k} x_i = t$ and $x_i \geq s$ for all i, then we have*

$$\sum_{i=0}^{k} x_i^2 \leq (t - ks)^2 + ks^2, \tag{3.1}$$

where the equality of (3.1) holds if and only if all x_i are identically s but one equals to $t - ks$.

Proof. We shall use induction on k to prove this result. It is obvious when $k = 0$. For $k \geq 1$, $\sum_{i=0}^{k} x_i = t$ implies that $\sum_{i=0}^{k-1} x_i = t - x_k$. By the induction hypothesis, we obtain that

$$\begin{aligned} \sum_{i=0}^{k} x_i^2 &\leq x_k^2 + (t - x_k - (k-1)s)^2 + (k-1)s^2 && (3.2) \\ &= \frac{1}{2}[2x_k - t + (k-1)s]^2 + \frac{1}{2}[t - (k-1)s]^2 + (k-1)s^2. && (3.3) \end{aligned}$$

Notice that $\sum_{i=0}^{k} x_i = t$ and $x_i \geq s$ for all i. Then we have $s \leq x_k \leq t - ks$. Thus,

$$-t + (k+1)s \leq 2x_k - t + (k-1)s \leq t - (k+1)s. \tag{3.4}$$

Applying (3.4) to (3.3), we can obtain that

$$\sum_{i=0}^{k} x_i^2 \leq \frac{1}{2}[t-(k+1)s]^2 + \frac{1}{2}[t-(k-1)s]^2 + (k-1)s^2 = (t-ks)^2 + ks^2.$$

Note that the equality of (3.1) holds if and only if the equalities of both (3.2) and (3.4) hold simultaneously. Clearly, the equality of (3.4) holds if and only if $x_k = s$ or $t - ks$. If $x_k = t - ks$, then it is easy to obtain that $x_i = s$ for $i \leq k-1$, since $\sum_{i=0}^{k} x_i = t$ and $x_i \geq s$ for all i. Meanwhile, the equality of (3.2) holds in this case. If $x_k = s$, by the induction hypothesis, the equality of (3.2) holds if and only if all x_i are identically s but one for all $i \leq k-1$. This completes the proof. □

Lemma 3.2. *If a tree T is L-DS, so is the product $T \times K_1$.*

Proof. To prove $T \times K_1$ is L-DS, assume that G is a graph L-cospectral to $T \times K_1$. We need to prove that G is isomorphic to $T \times K_1$. If $|VT| = n$, by Lemma 2.2, G is a connected graph with $|VG| = n+1$. By Lemmas 2.3 and 2.5, G can be written as the product of two graphs, then say $G = G_1 \times G_2$. Fix the following notations,

$$v_1 = |VG_1|, \quad e_1 = |EG_1|, \quad e_2 = |EG_2|.$$

Without loss of generality, we assume $|VG| \geq 2|VG_1|$, i.e., $n+1 \geq 2v_1$. Counting the edges of both G and $T \times K_1$ and applying Lemma 2.2, we obtain $e_1 + e_2 + v_1(n+1-v_1) = 2n-1$. It follows that

$$e_1 + e_2 = (2 - v_1)n + v_1^2 - v_1 - 1. \tag{3.5}$$

From Lemma 2.4, we only need to show that $v_1 = 1$, viz. $G = K_1 \times G_2$. Now suppose $v_1 \geq 2$. Applying $n+1 \geq 2v_1$ and $v_1 \geq 2$ to (3.5), we have

$$e_1 + e_2 \leq (2 - v_1)(2v_1 - 1) + v_1^2 - v_1 - 1 = -(v_1 - 1)(v_1 - 3). \tag{3.6}$$

Note that $e_1 + e_2 \geq 0$. It forces $v_1 = 2$ or 3. Then our proof will be complete with the following cases.

Case 1. $v_1 = 2$. Equation (3.5) implies $e_1 + e_2 = 1$. Then we have $e_1 = 1$ or $e_1 = 0$.

Case 1.1. $e_1 = 1$. Since $v_1 = 2$, it is easily seen that $G_1 = K_2 = K_1 \times K_1$. It follows that $G = G_1 \times G_2 = K_1 \times (K_1 \times G_2)$. Since G is L-cospectral to $T \times K_1$, applying Lemma 2.4, we have $G = T \times K_1$.

Case 1.2. $e_1 = 0$. Applying $v_1 = 2$ and $e_1 + e_2 = 1$, we can easily obtain that $G_1 = 2K_1$ and $G_2 = (n-3)K_1 + P_2$. Since $G = G_1 \times G_2$, by routine calculations, we have $\mathrm{Spec}(G_2) = \{2, [0]^{n-2}\}$. Applying Lemma 2.3, we have

$$\mathrm{Spec}(G) = \{n+1, n-1, 4, [2]^{n-3}, 0\}.$$

Since $\mathrm{Spec}(T \times K_1) = \mathrm{Spec}(G)$, by Lemma 2.3, the Laplacian spectrum of T is

$$\mathrm{Spec}(T) = \{n-2, 3, [1]^{n-3}, 0\}.$$

By Lemma 2.1, the number of spanning trees of T is given by $s(T) = \frac{3(n-2)}{n}$. But obviously $s(T) = 1$. It follows that $n = 3$. Hence, $G_2 = P_2$, and then $G = 2K_1 \times P_2 = K_1 \times P_3$. Now we can complete this case easily by applying Lemma 2.4.

Case 2. $v_1 = 3$. Equation (3.6) implies $e_1 = e_2 = 0$. Applying $v_1 = 3$ and $e_1 = e_2 = 0$ to (3.5), we can obtain $n = 5$. It follows that $G_1 = 3K_1$ and $G_2 = 3K_1$, and then $G = 3K_1 \times 3K_1$. Its Laplacian spectrum is $\{6, [3]^4, 0\}$. Since $\mathrm{Spec}(T \times K_1) = \mathrm{Spec}(G)$, by Lemma 2.3, the Laplacian spectrum of T is $\{[2]^4, 0\}$. Apply Lemma 2.1, we have $s(T) = \frac{16}{5}$, which is a contradiction. □

Theorem 3.3. *If a tree T is L-DS, so is the product $T \times K_m$ for all positive integers m.*

Proof. Suppose the graph G is L-cospectral to $T \times K_m$. We shall use induction on m to show that $G = T \times K_m$. The case $m = 1$ is stated in Lemma 3.2. Now we assume $m \geq 2$. Note that

$$T \times K_m = T \times \underbrace{K_1 \times \cdots \times K_1}_{m}.$$

Since $\mathrm{Spec}(G) = \mathrm{Spec}(T \times K_m)$, by Lemma 2.5, G is the product of $m+1$ graphs, denoted

$$G = G_0 \times G_1 \times \cdots \times G_m.$$

Fix notations as follows,

$$n = |VT|, \quad e_i = |EG_i|, \quad v_i = |VG_i| \quad \text{for} \quad i = 0, 1, \ldots, m. \tag{3.7}$$

Without loss of generality, assume $v_0 \geq v_1 \geq \cdots \geq v_m$. It is obvious that $\sum_{i=0}^m v_i = n + m$ by Lemma 2.2. In the following, we are going to prove $v_m = 1$ by contradiction. Now suppose $v_m \geq 2$. It follows that $v_i \geq 2$ for all $i = 0, \ldots, m$. Then we have $m + n = \sum_{i=0}^m v_i \geq 2(m+1)$, so $n \geq m+2$. For convenience, we list those conclusions as follows,

$$m \geq 2, \quad v_0 \geq \cdots \geq v_m \geq 2, \quad m + n = \sum_{i=0}^m v_i, \quad n \geq m+2. \tag{3.8}$$

Combining $v_0 \geq \cdots \geq v_m \geq 2$ with $\sum_{i=0}^m v_i = n + m$, by Proposition 3.1, we have

$$\sum_{i=0}^m v_i^2 \leq (n-m)^2 + 4m. \tag{3.9}$$

Since $\mathrm{Spec}(G) = \mathrm{Spec}(T \times K_m)$, Lemma 2.2 implies that G and $T \times K_m$ have the same number of edges. Counting the edges of both G and $T \times K_m$, we have

$$\sum_{i=0}^m e_i + \sum_{0 \leq i < j \leq m} v_i v_j = n - 1 + mn + \frac{m(m-1)}{2}. \tag{3.10}$$

Since $\sum_{i=0}^m v_i = n + m$, we have

$$\sum_{0 \leq i < j \leq m} v_i v_j = \frac{1}{2}\left(\left(\sum_{i=0}^m v_i\right)^2 - \sum_{i=0}^m v_i^2\right) = \frac{1}{2}\left((n+m)^2 - \sum_{i=0}^m v_i^2\right), \tag{3.11}$$

Applying (3.11) to (3.10), we obtain

$$\sum_{i=0}^{m} e_i = \frac{1}{2}\left(\sum_{i=0}^{m} v_i^2 - n^2 - m\right) + n - 1. \tag{3.12}$$

Applying (3.9) to (3.12), we have

$$\sum_{i=0}^{m} e_i \leq (1-m)n + \frac{1}{2}(m^2+3m) - 1. \tag{3.13}$$

Applying $m \geq 2$ and $n \geq m+2$ of (3.8) to (3.13), we have

$$\sum_{i=0}^{m} e_i \leq -\frac{1}{2}(m^2-m-2). \tag{3.14}$$

Notice that $-\frac{1}{2}(m^2-m-2) \leq 0$ for $m \geq 2$, but $\sum_{i=0}^{m} e_i \geq 0$. It follows that

$$m = 2, \quad e_i = 0 \quad \text{ for } \; i = 0, 1, 2. \tag{3.15}$$

Combining (3.15), (3.13), and $n \geq m+2$ of (3.8), we obtain $n = 4$. So far, we have obtained that $G = G_0 \times G_1 \times G_2$ satisfies

$$|VG_0| \geq |VG_1| \geq |VG_2| \geq 2, \quad |EG_0| = |EG_1| = |EG_2| = 0, \quad \text{and} \quad |VG| = m + n = 6.$$

It follows that

$$G = 2K_1 \times 2K_1 \times 2K_1$$

Then we have $\mathrm{Spec}(G) = \{[6]^2, [4]^3, 0\}$. Since $\mathrm{Spec}(G) = \mathrm{Spec}(T \times K_m)$, applying Lemma 2.3, we have $\mathrm{Spec}(T) = \{[2]^3, 0\}$. By Lemma 2.1, the number of spanning trees of T is $s(T) = 2$. Note the fact that T is a tree. It is a contradiction. Now we have shown that $v_m = 1$, and then $G_m = K_1$. From $\mathrm{Spec}(T \times K_m) = \mathrm{Spec}(G)$, we have

$$\mathrm{Spec}(K_1 \times (T \times K_{m-1})) = \mathrm{Spec}(K_1 \times (G_0 \times \cdots \times G_{m-1})).$$

By Lemma 2.3, we have

$$\mathrm{Spec}(T \times K_{m-1}) = \mathrm{Spec}(G_0 \times \cdots \times G_{m-1}).$$

By the induction hypothesis of $m-1$,

$$T \times K_{m-1} = G_0 \times \cdots \times G_{m-1}.$$

Thus $T \times K_m = G_0 \times \cdots \times G_m = G$. The proof is complete. □

Remark 3.1. *Up until now, there are so many trees are proved to be L-DS graphs, for examples, path P_n [12], graphs Z_n, T_n, and W_n [11], starlike tree S [10], etc. Therefore, Theorem 3.3 implies that $P_n \times K_m$, $Z_n \times K_m$, $T_n \times K_m$, $W_n \times K_m$ and $S \times K_m$ are also L-DS graphs.*

4 Laplacian spectral characterization of the products of unicyclic graphs and complete graphs

This section is devoted to the Laplacian spectral characterization of the products of unicyclic graphs and complete graphs. Recall that a unicyclic graph is a connected graph containing exactly one cycle. In other words, a connected graph $G = (VG, EG)$ is unicyclic iff $|VG| = |EG|$. In notations, we write the unicyclic graph as U. For $k \leq n$, denote by $\mathcal{U}(n, k)$ the collection of all unicyclic graphs U with $|VU| = n$ and containing the cycle C_k as a subgraph. Recall in Lemma 2.6 that given a vertex v of the graph G, $d(v)$ denotes the degree of v, and $m(v)$ is defined to be $m(v) = \dfrac{1}{d(v)} \sum_{\{u,v\} \in EG} d(u)$.

Proposition 4.1. *With above notations, for all $U \in \mathcal{U}(n,k)$, we have*

$$\max\{d(v)+m(v) \mid v \in VU\} \leq n-k+3+\frac{2}{n-k+2}. \tag{4.1}$$

The equality of (4.1) *holds if and only if U is the graph obtained by appending $n-k$ vertices to a vertex of the cycle C_k.*

Proof. Since $U \in \mathcal{U}(n,k)$ contains the cycle C_k as a subgraph, then $|VU \setminus VC_k| = n-k$. It is easily seen that the maximum vertex degree of U is $n-k+2$, viz.

$$d(v) \leq n-k+2 \quad \text{for all} \quad v \in VU. \tag{4.2}$$

Given $v_0 \in VU$, we shall prove (4.1) by studying the following cases of $d(v_0)$.

Case 1. $d(v_0)=1$. Clearly, $v_0 \notin VC_k$ and there is a unique vertex adjacent to v_0, denoted $v \in VU$. By (4.2), we have $d(v) \leq n-k+2$. Thus

$$d(v_0)+m(v_0) = d(v_0)+d(v) \leq n-k+3.$$

Case 2. $n-k+2 \geq d(v_0) \geq 2$. To prove (4.1), viz. to find the maximum value $m(v_0)$ for v_0 with fixed $d(v_0)$, it is enough to find the maximum value of the sum

$$\sum_{\{v,v_0\}\in EG} d(v). \tag{4.3}$$

Note that U is unicyclic with the cycle C_k. Consider the following vertex set

$$V_0 = \{u \in VU \setminus VC_k \mid u \text{ is not adjacent to } v_0\}.$$

Since U is unicyclic, v_0 has at most two neighbors in VC_k. And v_0 has two neighbors in VC_k occurs only when $v_0 \in VC_k$. It implies that

$$|V_0| \leq n-k-d(v_0)+2. \tag{4.4}$$

In order to make the sum (4.3) as large as possible, assume that all vertices of V_0 are adjacent to neighbors of v_0. Now, the sum (4.3) equals

$$n-k-d(v_0)+2+d(v_0)+2 = n-k+4.$$

Clearly, this is the maximum value for the sum (4.3). Thus, in general, we have

$$\sum_{\{v,v_0\}\in EU} d(v) \leq n-k+4.$$

It follows that

$$d(v_0)+m(v_0) \leq d(v_0)+\frac{n-k+4}{d(v_0)}.$$

Now we are going to find an upper bound of $d(v_0)+\frac{n-k+4}{d(v_0)}$ with $2 \leq d(v_0) \leq n-k+2$. Note that the maximum value of $d(v_0)+\frac{n-k+4}{d(v_0)}$ occurs only when $d(v_0)=2$ or $n-k+2$. On the other hand, to compare these two values, we have

$$\left(n-k+2+\frac{n-k+4}{n-k+2}\right)-\left(2+\frac{n-k+4}{2}\right) = \frac{n-k+2}{2}+\frac{2}{n-k+2}-2 \geq 0.$$

It is easily seen that $d(v_0)+\frac{n-k+4}{d(v_0)}$ is maximum iff $d(v_0)=n-k+2$. Hence,

$$d(v_0)+m(v_0) \leq n-k+3+\frac{2}{n-k+2},$$

where the equality holds iff $d(v_0)=n-k+2$. Note that $d(v_0)=n-k+2$ implies that $V_0=\emptyset$ by (4.4). This completes the proof. □

Lemma 4.2. *If a unicyclic graph U is L-DS and $U \neq C_6$, then $U \times K_1$ is L-DS. Moreover, $C_6 \times K_1$ is L-cospectral to $2K_1 \times (2P_2 + K_1)$, see Figure 1.*

Proof. The idea of the proof is almost the same as Lemma 3.2. Similarly, assume that G is a graph L-cospectral to $U \times K_1$. We shall determine the condition, under which G is isomorphic to $U \times K_1$. Let $|VU| = n$, by Lemma 2.2, then G is a connected graph with $|VG| = n + 1$. By Lemmas 2.3 and 2.5, G can be written as the product of two graphs G_1 and G_2, i.e., $G = G_1 \times G_2$. Fix the following notations,

$$v_1 = |VG_1|, \quad e_1 = |EG_1|, \quad e_2 = |EG_2|.$$

Without loss of generality, we assume $|VG| \geq 2|VG_1|$, i.e., $n + 1 \geq 2v_1$. Counting the edges of both G and $U \times K_1$ and applying Lemma 2.2, we obtain $e_1 + e_2 + v_1(n + 1 - v_1) = 2n$. It follows that

$$e_1 + e_2 = (2 - v_1)n + v_1^2 - v_1. \tag{4.5}$$

From Lemma 2.4, it would be enough if we obtain $v_1 = 1$, viz. $G = K_1 \times G_2$. Now suppose $v_1 \geq 2$. Applying $n + 1 \geq 2v_1$ and $v_1 \geq 2$ to (4.5), we have

$$e_1 + e_2 \leq (2 - v_1)(2v_1 - 1) + v_1^2 - v_1 = -(v_1 - 2)^2 + 2. \tag{4.6}$$

Notice the fact $e_1 + e_2 \geq 0$. It forces $v_1 = 2$ or 3. Then our proof will be complete with the following cases.

Case 1. $v_1 = 2$. Applying $v_1 = 2$ to (4.5), we have $e_1 + e_2 = 2$. Notice that $v_1 = |VG_1| = 2$ implies $e_1 = |EG_1| \leq 1$. Then $e_1 = 1$ or 0.

Case 1.1. $e_1 = 1$. Since $v_1 = 2$, it is clear that $G_1 = K_2$. Then we have

$$G = K_2 \times G_2 = K_1 \times (K_1 \times G_2).$$

Since $\mathrm{Spec}(U \times K_1) = \mathrm{Spec}(G)$ and U is L-DS, by Lemma 2.4, we obtain that $K_1 \times G_2$ and U are isomorphic. Clearly, $G = U \times K_1$.

Case 1.2. $e_1 = 0$. It is clear that $G_1 = 2K_1$. Since $e_1 + e_2 = 2$, then we have $e_2 = |EG_2| = 2$. Depending on two edges of G_2 either adjacent or not, G_2 may be isomorphic to $P_3 + (n - 4)K_1$ or $2P_2 + (n - 5)K_1$. Thus, we have

$$G = 2K_1 \times (P_3 + (n - 4)K_1)\,, \text{ or } G = 2K_1 \times (2P_2 + (n - 5)K_1)\,.$$

Case 1.2.1. $G = 2K_1 \times (P_3 + (n - 4)K_1)$. Since $\mathrm{Spec}(P_3) = \{3, 1, 0\}$, applying Lemma 2.3, we have

$$\mathrm{Spec}(G) = \{n + 1, n - 1, 5, 3, [2]^{n-4}, 0\}.$$

Since $\mathrm{Spec}(G) = \mathrm{Spec}(U \times K_1)$, applying Lemma 2.3 again, we obtain

$$\mathrm{Spec}(U) = \{n - 2, 4, 2, [1]^{n-4}, 0\}.$$

By Lemma 2.1, the number of the spanning trees of U is $s(U) = \frac{8(n-2)}{n}$. It forces $n = 4, 8$ or 16.

Case 1.2.1.1. $n = 4$. Clearly, we have $G = 2K_1 \times P_3$. Notice that the unicyclic graph U has $s(U) = \frac{8(n-2)}{n} = 4$ spanning trees. It follows that the cycle of U is C_4. But $|VU| = n = 4$, then $U = C_4$. On the other hand, it is easily seen that $2K_1 \times P_3 = K_1 \times C_4$, viz. $G = K_1 \times U$ in this case.

Case 1.2.1.2. $n = 8$. Since $s(U) = \frac{8(n-2)}{n} = 6$ and U is unicyclic, then the cycle C_6 is a subgraph of U, i.e., $U \in \mathcal{U}(8, 6)$. By Proposition 4.1, we have $d(v) + m(v) \leq 5.5$ for all $v \in VU$. Notice that the maximum Laplacian eigenvalue of U is $\lambda(U) = n - 2 = 6$. It is a contradiction by Lemma 2.6.

Case 1.2.1.3. $n = 16$. Similar as *Case 1.2.1.2*, we have $s(U) = 7$ and $U \in \mathcal{U}(16, 7)$. By Proposition 4.1, $d(v) + m(v) \leq 12 + \frac{2}{11}$ for all $v \in VU$. So it is a contradiction by Lemma 2.6 since the maximum Laplacian of U is $\lambda(U) = n - 2 = 14$ for the current case.

Case 1.2.2. $G = 2K_1 \times (2P_2 + (n-5)K_1)$. Similar as *Case 1.2.1*, applying Lemma 2.3, we obtain that

$$\mathrm{Spec}(G) = \{n+1, n-1, [4]^2, [2]^{n-4}, 0\}, \text{ and } \mathrm{Spec}(U) = \{n-2, [3]^2, [1]^{n-4}, 0\}.$$

It follows that the number of spanning trees of U is $s(U) = \frac{9(n-2)}{n}$, so $n = 6$, 9, or 18.

Case 1.2.2.1. $n = 6$. Similar as *Case 1.2.1.2*, we have $U \in \mathcal{U}(6,6)$, which implies U is exactly the cycle C_6. By routine calculations, we can check that

$$\mathrm{Spec}(K_1 \times C_6) = \mathrm{Spec}(2K_1 \times (2P_2 + K_1)).$$

But it is easily seen that $K_1 \times C_6$ and $2K_1 \times (2P_2 + K_1)$ are not isomorphic, see Figure 1.

Case 1.2.2.2. $n = 9$. Similar as *Case 1.2.1.2*, we have $U \in \mathcal{U}(9,7)$, and then $d(v) + m(v) \leq 5.5$ for all $v \in VU$. But $\lambda(U) = 7$ in this case, so it is a contradiction by Lemma 2.6.

Case 1.2.2.3. $n = 18$. The arguments in this case is also similar as *Case 1.2.1.2*. It is a contradiction by Lemma 2.6 since $U \in \mathcal{U}(18,8)$ and $m(v) + d(v) \leq 13 + \frac{1}{6}$ for all $v \in VU$, but $\lambda(U) = 16$.

Case 2. $v_1 = 3$. Applying $v_1 = 3$ to (4.5), we have $e_1 + e_2 = 6 - n$. Using the fact $e_1 + e_2 \geq 0$, then $n \leq 6$. Notice that $|VG| = n + 1 \geq 2|VG_1| = 6$ implies $n \geq 5$. Thus, $n = 5$ or 6.

Case 2.1. $n = 6$. Applying $v_1 = 3$ and $n = 6$ to (4.5), we can obtain $e_1 + e_2 = 0$, viz. $e_1 = e_2 = 0$. Then we have $G = 3K_1 \times 4K_1$, whose Laplacian spectrum is $\mathrm{Spec}(G) = \{7, [4]^2, [3]^3, 0\}$. Since $\mathrm{Spec}(G) = \mathrm{Spec}(U \times K_1)$, applying Lemma 2.3, we have $\mathrm{Spec}(U) = \{[3]^2, [2]^3, 0\}$. Using Lemma 2.1, we have $s(U) = 12$. However, $|VU| = n = 6$, it follows that $s(U) \leq 6$, a contradiction.

Case 2.2. $n = 5$. Our arguments are similar as *Case 2.1*. When $n = 5$, we have $e_1 + e_2 = 1$, and then $G = (P_2 + K_1) \times 3K_1$. Similar as *Case 2.1*, we can easily obtain $s(U) = \frac{32}{5}$. Note the fact that $s(U)$ is an integer, a contradiction. This completes the proof. $\square$

The following result is obvious from Lemmas 4.2 and 2.3. Indeed, L-cospectral graphs shown in Figure 1 have also been given in [15].

Corollary 4.3. *If $n \neq 6$, then $C_n \times K_m$ is L-DS for all $m \geq 1$. Moreover, the L-cospectral graph of $C_6 \times K_m$ is $H_1 \times K_{m-1}$, where $H_1 = 2K_1 \times (2P_2 + K_1)$ as shown in Figure 1.*

Theorem 4.4. *If a unicyclic graph U is L-DS and $U \neq C_6$, then the product $U \times K_m$ is L-DS for all positive integers m.*

Proof. The idea to prove this theorem is similar as the proof of Theorem 3.3. In the following, we borrow all of arguments and notations ahead of (3.10) in the proof of Theorem 3.3, except that the tree T is replaced by the unicyclic graph U. In the following, we prove the theorem by induction on m. Note that the case $m = 1$ is Lemma 4.2. Now assume $m \geq 2$. We are going to prove $v_m = 1$ by contradiction. Suppose $v_m \geq 2$. Since $|EU| = |VU| = n$, instead of (3.10), we have

$$\sum_{i=0}^{m} e_i + \sum_{0 \leq i < j \leq m} v_i v_j = n + mn + \frac{m(m-1)}{2}. \tag{4.7}$$

Then by the same arguments as Theorem 3.3, instead of (3.12), (3.13), and (3.14), we have

$$\sum_{i=0}^{m} e_i = \frac{1}{2}\left(\sum_{i=0}^{m} v_i^2 - n^2 - m\right) + n; \tag{4.8}$$

$$\sum_{i=0}^{m} e_i \leq (1-m)n + \frac{1}{2}(m^2 + 3m); \tag{4.9}$$

$$\sum_{i=0}^{m} e_i \leq -\frac{1}{2}(m^2 - m - 4). \tag{4.10}$$

Note that $\sum_{i=0}^{m} e_i \geq 0$. Applying the assumption $m \geq 2$ to (4.10), we have $m = 2$. Then (4.9) becomes

$$e_0 + e_1 + e_2 \leq -n + 5.$$

It follows that $n \leq 5$. On the other hand, following the arguments of Theorem 3.3, we also have, similar as (3.8), $n \geq m + 2 = 4$. Combining them together, we have $n = 4$ or 5.

Case 1. $n = 4$. It is easily obtained that $v_0 + v_1 + v_2 = m + n = 6$. Recall the assumption $v_0 \geq v_1 \geq v_2 \geq 2$. It follows that $v_0 = v_1 = v_2 = 2$. Now applying $m = 2$, $n = 4$, and $v_0 = v_1 = v_2 = 2$ to (4.8), we have $e_0 + e_1 + e_2 = 1$. It is easily seen that

$$G = K_2 \times 2K_1 \times 2K_1 = C_4 \times K_2.$$

Since $\mathrm{Spec}(G) = \mathrm{Spec}(U \times K_2)$, by Lemma 2.4, we have $U = C_4$, and then $G = U \times K_2$.

Case 2. $n = 5$. Clearly, $v_0 + v_1 + v_2 = 7$. Since $v_0 \geq v_1 \geq v_2 \geq 2$, then we have

$$v_0 = 3, \quad v_1 = 2, \quad v_2 = 2.$$

Applying these to (4.8), we obtain $e_0 = e_1 = e_2 = 0$. It means that

$$G = 3K_1 \times 2K_1 \times 2K_1.$$

From Lemma 2.3, by routine calculations, we have

$$\mathrm{Spec}(G) = \{[7]^2, [5]^2, [4]^2, 0\}.$$

Since $\mathrm{Spec}(G) = \mathrm{Spec}(U \times K_2)$, applying Lemma 2.3 again, we have

$$\mathrm{Spec}(U) = \{[3]^2, [2]^2, 0\}.$$

It is a contradiction since the number of spanning tree of U is $s(U) = \frac{36}{5}$ by Lemma 2.1. So far, what we have obtained is $v_m = 1$, i.e., $G_m = K_1$. Since $\mathrm{Spec}(G) = \mathrm{Spec}(U \times K_m)$, namely,

$$\mathrm{Spec}((G_0 \times \cdots \times G_{m-1}) \times K_1) = \mathrm{Spec}((U \times K_{m-1}) \times K_1),$$

by Lemma 2.3, it is easy to obtain that

$$\mathrm{Spec}(G_0 \times \cdots \times G_{m-1}) = \mathrm{Spec}(U \times K_{m-1}).$$

From the induction hypothesis of $m - 1$, we have

$$G_0 \times \cdots \times G_{m-1} = U \times K_{m-1}.$$

Obviously, we have $G = U \times K_m$. This completes the proof. □

Up until now, there are only few unicyclic graphs have been proved to be L-DS graphs. For example, *lollipop graph*, which is a graph obtained by attaching a pendant vertex of a path to a cycle, and graph $H(n; q, n_1, n_2)$ with order n, which contains a cycle C_q and two hanging paths P_{n_1} and P_{n_2} attached at the same vertex of the cycle, are proved to be L-DS graph [3, 6]. Thus we can trivially get the following results.

Corollary 4.5. *Let G be the lollipop graph. Then $G \times K_m$ is L-DS for all positive integers m.*

Corollary 4.6. *Let $G = H(n; q, n_1, n_2)$. Then $G \times K_m$ is L-DS for all positive integers m.*

5 Laplacian spectral characterization of the products of bicyclic graphs and complete graphs

Recall that a bicyclic graph is a connected graph with two independent cycles. Strictly speaking, a connected graph $G = (VG, EG)$ is called bicyclic if $|EG| = |VG| + 1$. From now on, we shall denote by B the bicyclic graph to distinguish it from other graphs. This section is devoted to the study of Laplacian spectral characterization of the products of bicyclic graphs and complete graphs. The main result is that the products $B \times K_m$ are L-DS for all L-DS bicyclic graphs B but one, $B = \Theta_{3,2,5}$, see Figure 2. Before proceeding, we need some preparations.

Let $G_1 = (VG_1, EG_1)$ and $G_2 = (VG_2, EG_2)$ be two connected graphs. The *union* of G_1 and G_2 is defined to be

$$G_1 \cup G_2 = (VG_1 \cup VG_2, EG_1 \cup EG_2),$$

and the *intersection* of G_1 and G_2 is defined to be

$$G_1 \cap G_2 = (VG_1 \cap VG_2, EG_1 \cap EG_2).$$

Let C_r and C_s be two cycles. If $C_r \cap C_s$ is a path P_t of size $t \geq 1$, then the graph union $C_r \cup C_s$ is denoted by $\Theta_{r,t,s}$. If $C_r \cap C_s$ is the empty graph, but C_r and C_s is connected by a path, then the union of C_r and C_s, together with the path between them, is denoted by $\Theta_{r,0,s}$. The subscription t and 0 represent the size of the intersection of two cycles. Graphs $\Theta_{r,t,s}$ for all $0 \leq t \leq r \leq s$ are called Θ-*graphs.* Informally speaking, the Θ-graph is a bicyclic graph, either consisting of two cycles whose intersection is a path, or obtained by appending two disjoint cycles to two ends of a path. Note that the graph $\Theta_{r,t,s}$ with $2 \leq t \leq r \leq s$, are the same as graphs $\Theta_{r,r-t+2,r+s-2t+2}$ and $\Theta_{s,s-t+2,r+s-2t+2}$. For example, $\Theta_{3,2,4} = \Theta_{4,4,5} = \Theta_{3,3,5}$. To avoid this situation, we set $r, s \geq 2(t-1)$ for $t \geq 2$. Then we assume that

$$s \geq r \geq 2t - 2 \quad \text{for} \quad t \geq 0. \tag{5.1}$$

Clearly, a graph B is bicyclic iff B contains exactly one Θ-graph. Denote by $\mathcal{B}(n, r, t, s)$ the collection of bicyclic graphs B with $\Theta_{r,t,s}$ as a subgraph and $|VB| = n$. For $B \in \mathcal{B}(n, r, t, s)$, since

$$|VB| \geq |V\Theta_{r,t,s}| \geq r + s - t \quad \text{for} \quad t \geq 0,$$

then we have

$$n \geq r + s - t. \tag{5.2}$$

Now, we give one proposition which will play an important rule in this section.

Proposition 5.1. *Let $B \in \mathcal{B}(n, r, t, s)$ be a bicyclic graph, and denote $\alpha(B)$ the following number*

$$\alpha(B) = \max\{d(v) + m(v) \mid v \in VB\}.$$

(1) *If $t = 0$, then*

$$s(B) = rs, \quad d(v) \leq n - r - s + 3, \quad \alpha(B) = n - r - s + 4 + \frac{4}{n - r - s + 3}. \tag{5.3}$$

(2) *If $t = 1$, then*

$$s(B) = rs, \quad d(v) \leq n - r - s + 5, \quad \alpha(B) = n - r - s + 6 + \frac{4}{n - r - s + 5}. \tag{5.4}$$

(3) *If $t = 2$, then*

$$s(B) = rs - 1, \quad d(v) \leq n - r - s + 5, \quad \alpha(B) = n - r - s + 6 + \frac{4}{n - r - s + 5}. \tag{5.5}$$

(4) *If $t \geq 3$, then*

$$s(B) = rs - (t-1)^2, \quad d(v) \leq n + t - r - s + 3, \quad \alpha(B) = n + t - r - s + 4 + \frac{3}{n + t - r - s + 3}. \tag{5.6}$$

Proof. In the following, we just give a detailed proof for the case $t = 0$. The other cases will be followed by similar arguments.

Assume $B \in \mathcal{B}(n, r, 0, s)$. B contains $\Theta_{r,0,s}$, and then two disjoint cycles C_r and C_s. Each spanning tree of B can be obtained by removing two edges from B, that is, one from EC_r and the other from EC_s. It implies that the number of spanning trees of B is $s(B) = rs$.

To prove $d(v) \leq n - r - s + 3$ for all $v \in VB$, we denote by $\Theta'_{r,0,s}$ the Θ-graph $\Theta_{r,0,s}$, whose two cycles C_r and C_s are connected by the path P_2. It is clear that any $\Theta_{r,0,s}$ contains at least $r + s$ vertices. Note that $|V\Theta_{r,0,s}| \geq r + s$ and the equality holds iff $\Theta_{r,0,s} = \Theta'_{r,0,s}$. Then we have

$$|VB \setminus V\Theta_{r,0,s}| = |VB| - |V\Theta_{r,0,s}| \leq n - r - s.$$

In order to make the maximum vertex degree of B is $n - r - s + 3$, B must be obtained from $\Theta'_{r,0,s}$ by attaching $n - r - s$ vertices to a vertex with degree 3 in $\Theta'_{r,0,s}$. Clearly, this is the maximum case.

Now we are ready to prove the last equality of (5.3). If $t = 0$, to make $m(v) + d(v)$ maximum, it is clearly required that B contains $\Theta'_{r,0,s}$ as a subgraph. Suppose $\alpha(B) = d(v_0) + m(v_0)$ for some $v_0 \in VB$. This easily implies that v_0 must be an end of the path P_2 connecting two cycles of $\Theta'_{r,0,s}$. Then we have $3 \leq d(v_0) \leq n - r - s + 3$. Denote by $n(v) = \sum_{\{u,v\} \in EG} d(u)$ the total degree of neighbors of v. Then we have

$$\alpha(B) = \max \left\{ d(v_0) + \frac{n(v_0)}{d(v_0)} \mid 3 \leq d(v_0) \leq n - r - s + 3 \right\}.$$

To make $d(v_0) + \frac{n(v_0)}{d(v_0)}$ as large as possible, all vertices in $VB \setminus V\Theta'_{r,0,s}$ are, either incident with v_0, or incident with neighbors of v_0. It follows that $n(v_0)$ is a constant $n - r - s + 7$. By some arithmetic calculations, we can obtain that, for $d(v_0) = n - r - s + 3$,

$$\alpha(B) = n - r - s + 3 + \frac{n - r - s + 7}{n - r - s + 3} = n - r - s + 4 + \frac{4}{n - r - s + 3}.$$

This completes the proof of case $t = 0$. $\square$

Lemma 5.2. *If a bicyclic graph B is L-DS and $B \neq \Theta_{3,2,5}$, then $B \times K_1$ is L-DS. Furthermore, $\Theta_{3,2,5} \times K_1$ is L-cospectral to $2K_1 \times (P_4 + K_1)$, see Figure 2.*

Proof. We shall use similar arguments and notations as Lemma 3.2 or 4.2. Then we can assume that $G = G_1 \times G_2$ is a graph L-cospectral to $B \times K_1$ and

$$|VB| = n, \quad |VG_1| = v_1, \quad |VG_2| = v_2, \quad |EG_1| = e_1, \quad |EG_2| = e_2.$$

We further assume $n + 1 \geq 2v_1$. By counting the edges of both $G_1 \times G_2$ and $B \times K_1$, Lemma 2.2 implies

$$e_1 + e_2 = (2 - v_1)n + v_1^2 - v_1 + 1. \tag{5.7}$$

By Lemma 2.4, we are required to prove $v_1 = 1$. Suppose $v_1 \geq 2$, applying $n + 1 \geq 2v_1$ to (5.7), we have

$$e_1 + e_2 \leq (2 - v_1)(2v_1 - 1) + v_1^2 - v_1 + 1 = -v_1^2 + 4v_1 - 1. \tag{5.8}$$

The fact $e_1 + e_2 \geq 0$ implies that $v_1 = 2$ or 3.

Case 1. $v_1 = 2$. Clearly, $v_1 = 2$ implies that $e_1 \leq 1$. If $e_1 = 1$, the graph $G_1 = K_1 \times K_1$, and then G can be written as $K_1 \times (K_1 \times G_2)$, which is clear from Lemma 2.4. If $e_1 = 0$, then $G_1 = 2K_1$. Substituting $v_1 = 2$ into (5.7), we have $e_1 + e_2 = 3$. Then we obtain $e_2 = 3$. According to $v_2 = n - 1$ and $e_2 = 3$, G_2 has to be one of the following graphs

$$3P_2 + (n-7)K_1, \quad P_2 + P_3 + (n-6)K_1,$$
$$P_4 + (n-5)K_1, \quad K_1 \times 3K_1 + (n-5)K_1, \quad C_3 + (n-4)K_1.$$

Then consider the following cases.

Case 1.1. $G_2 = 3P_2 + (n-7)K_1$. Since $G_1 = 2K_1$, applying Lemma 2.3 and by routine calculations, we can obtain

$$\mathrm{Spec}(G) = \{n+1, n-1, [4]^3, [2]^{n-5}, 0\}.$$

It follows that

$$\mathrm{Spec}(B) = \{n-2, [3]^3, [1]^{n-5}, 0\}. \tag{5.9}$$

The number of spanning trees of B is given by

$$s(B) = 27 - \frac{54}{n}.$$

Thus $n \mid 54$. On the other hand, it is clear that $n \geq 7$ since $G_2 = 3P_2 + (n-7)K_1$. Hence, $n = 9, 18, 27$, or 54. In the following, we shall rule out them case by case.

Case 1.1.1. $n = 9$. From (5.9), the maximum Laplacian eigenvalue of B is 7. By Lemmas 2.1 and 2.6, it is easily obtained that

$$\alpha(B) \geq 7, \quad s(B) = 21.$$

Assume $B \in \mathcal{B}(9, r, t, s)$, i.e., B contains $\Theta_{r,t,s}$ as a subgraph. Then $r + s \leq 9 + t$ by (5.2). If $t = 0$, (5.3) implies $s(B) = rs = 21$ which is impossible for $r + s \leq 9$. If $t = 1$, (5.4) implies $rs = 21$. It follows that $r = 3, s = 7$. Then using (5.4) again, $\alpha(B) \leq 6$, a contradiction. If $t = 2$, (5.5) implies $s(B) = rs - 1 = 21$ which has no solution since $s \geq r \geq 3$. If $t \geq 3$, from (5.6), we have $\alpha(B) \leq n + t - r - s + 5$. Applying $r + s \geq 4t - 4$ in (5.1), we obtain $\alpha(B) \leq n + 9 - 3t$. Note that $\alpha(B) \geq n - 2$. Then we have the following inequality

$$n - 2 \leq \alpha(B) \leq n + 9 - 3t. \tag{5.10}$$

Combined with $t \geq 3$, it forces $t = 3$. From $s(B) = 21 = rs - (t-1)^2$, we obtain $r = s = 5$. Applying $n = 9, t = 3, r = s = 5$ to (5.6), it is easily obtained that $\alpha(B) \leq \frac{33}{5} < 7$, which is a contradiction. Hence, we proved $n \neq 9$

Case 1.1.2. $n = 18, 27$, or 54. We just give the case $n = 18$ in details, but skip the cases $n = 27, 54$, since all arguments are the same. Suppose that $n = 18$ and $B \in \mathcal{B}(18, r, t, s)$. Then the maximum Laplacian eigenvalue of B is $n - 2 = 16$. Lemmas 2.1 and 2.6 imply

$$\alpha(B) \geq 16, \quad s(B) = 24.$$

If $t = 0$ or 1, then $s(B) = rs = 24$. It implies that $r = 4, s = 6$ or $r = 3, s = 8$. Applying (5.3) and (5.4) to all cases of t, r, s, we obtain $\alpha(B) < 15$, contradictions. If $t = 2$, then $s(B) = rs - 1 = 24$, i.e., $r = s = 5$. By (5.5), we get $\alpha(B) < 15$, a contradiction. If $t \geq 3$, then (5.10) implies $t = 3$. Since $s(B) = rs - (t-1)^2 = 24$ and $s \geq r \geq 2t - 2$, we obtain $t = 3, r = 4, s = 7$. By (5.6), we have $\alpha(B) < 15$, a contradiction.

Case 1.2. $G_2 = P_2 + P_3 + (n-6)K_1$. Since $\mathrm{Spec}(P_3) = \{3, 1, 0\}$, by Lemma 2.3, we have

$$\mathrm{Spec}(G) = \{n+1, n-1, 5, 4, 3, [2]^{n-5}, 0\},$$

and then

$$\mathrm{Spec}(B) = \{n-2, 4, 3, 2, [1]^{n-5}, 0\}.$$

From Lemma 2.1, $s(B) = 24 - \frac{48}{n}$. Note that $n \geq 6$ for $G_2 = P_2 + P_3 + (n-6)K_1$. It follows that $n = 6, 8, 12, 16, 24$, or 48.

Case 1.2.1. $n = 6$. Then the maximum Laplacian eigenvalue of B is $n - 2 = 4$. Lemma 2.7 implies that the maximum vertex degree of B, denoted $\Delta(B)$, is at most 3 and not identical 3. Namely, all vertices degree of B is at most 2. It is a contradiction with that B is a bicyclic graph.

Case 1.2.2. $n = 8$. The maximum Laplacian eigenvalue of B is $n - 2 = 6$. Lemmas 2.1 and 2.6 imply that

$$\alpha(B) \geq 6, \quad s(B) = 18.$$

If $t = 0$ or 1, then $s(B) = rs = 18$. It follows that $r = 3, s = 6$. Then (5.3) and (5.4) imply $\alpha(B) < 6$, contradictions. If $t = 2$, then $s(B) = rs - 1 = 18$ which has no solutions of r, s. If $t \geq 3$, then (5.10) implies $t = 3$. It is impossible since $s(B) = rs - (t-1)^2 = 18$ and $s \geq r \geq 2t - 2 = 4$.

Case 1.2.3. $n = 12, 16, 24$, or 48. We only show $n \neq 12$ in details. The others are similar. Suppose $n = 12$. Then Lemmas 2.1 and 2.6 imply that

$$\alpha(B) \geq 10, \quad s(B) = 20.$$

If $t = 0$ or 1, then $s(B) = rs = 20$. It follows that $r = 4, s = 5$. (5.3) and (5.4) imply $\alpha(B) < 10$, contradictions. If $t = 2$, then $s(B) = rs - 1 = 20$, i.e., $r = 3, s = 7$. (5.5) implies $\alpha(B) < 9$, a contradiction. If $t \geq 3$, then (5.10) implies $t = 3$. Since $s(B) = rs - (t-1)^2 = 20$, we obtain $t = 3, r = 4, s = 6$. By (5.6), we have $\alpha(B) < 10$, a contradiction.

Case 1.3. $G_2 = P_4 + (n-5)K_1$. Then $n \geq 5$. By routine calculations as above, we can obtain

$$\mathrm{Spec}(B) = \{n-2, 3+\sqrt{2}, 3, 3-\sqrt{2}, [1]^{n-5}, 0\},$$

and then $s(B) = 21 - \dfrac{42}{n}$. It follows that n must be $6, 7, 14, 21$, or 42.

Case 1.3.1. $n = 6$. It follows that $s(B) = 14$. By similar arguments as above, we will obtain $t = 2, r = 3, s = 5$ which can not be ruled out. Then B consists of two cycles, C_3 and C_5, whose intersection is the path P_2, i.e., $B = \Theta_{3,2,5}$. Indeed, by routine calculations, we obtain that the Laplacian spectrum of $\Theta_{3,2,5}$ is exactly $\{3+\sqrt{2}, 4, 3-\sqrt{2}, 3, 1, 0\}$. Hence, $\Theta_{3,2,5} \times K_1$ is L-cospectral to $2K_1 \times (P_4 + K_1)$, but not isomorphic, see Figure 2.

Case 1.3.2. $n = 7$. Note that the maximum Laplacian eigenvalue of B is 5. Applying Lemmas 2.1, 2.6 and 2.7, we obtain

$$\alpha(B) \geq 5, \quad \Delta(B) \leq 3, \quad s(B) = 15,$$

where $\Delta(B)$ is the maximum vertex degree of B. Thus $t \neq 1$, otherwise, B has a vertex of degree at least 4. If $t = 0$, then (5.3) implies $s(B) = rs = 15$, a contradiction to (5.2). If $t = 2$, then (5.5) implies $s(B) = rs - 1 = 15$, i.e., $r = s = 4$. Since the maximum vertex degree of B is 3, combining with $n = 7, t = 2, r = s = 4$, we obtain that the degree sequence of B is $(3,3,3,2,2,2,1)$, denoted by

$$\deg(B) = ([3]^3, [2]^3, 1),$$

where $[a]^b$ is a sequence of constant a with multiplicity b. It follows that

$$\deg(B \times K_1) = (7, [4]^3, [3]^3, 2).$$

On the other hand, since $G_1 = 2K_1$ and $G_2 = P_4 + (n-5)K_1 = P_4 + 2K_1$, we have

$$\deg(G_1 \times G_2) = ([6]^2, [4]^2, [3]^2, [2]^2).$$

But it is easily checked that $7^2 + 3 \cdot 4^2 + 3 \cdot 3^2 + 2^2 \neq 2 \cdot (6^2 + 4^2 + 3^2 + 2^2)$, a contradiction to Lemma 2.2. If $t \geq 3$, then (5.10) implies $t = 3$. It is impossible since $rs = s(B) + (t-1)^2 = 19$.

Case 1.3.3. $n = 14, 21$, or 42. We only disprove $n = 42$ in details. The others are similar. Suppose $n = 42$. Lemmas 2.1 and 2.6 imply that

$$\alpha(B) \geq 40, \quad s(B) = 20.$$

If $t = 0$ or 1, then $s(B) = rs = 20$, and then $r = 4, s = 5$. Applying (5.3) and (5.4), we obtain $\alpha(B) < 40$, contradictions. If $t = 2$, then $s(B) = rs - 1 = 20$, i.e., $r = 3, s = 7$. (5.5) implies $\alpha(B) < 39$, a contradiction.

If $t \geq 3$, then (5.10) implies $t = 3$. Since $s(B) = rs - (t-1)^2 = 20$, we obtain $t = 3, r = 4, s = 6$. By (5.6), we have $\alpha(B) < 40$, a contradiction.

Case 1.4. $G_2 = K_1 \times 3K_1 + (n-5)K_1$. Similar as above, we have $n \geq 5$ and

$$\mathrm{Spec(B)} = \{n-2, 5, [2]^2, [1]^{n-5}, 0\},$$

and then the number of spanning trees of B is $s(B) = 20 - \frac{40}{n}$. It forces $n = 5, 8, 10, 20$, or 40.

Case 1.4.1. $n = 5$. Since $\mathrm{Spec}(B) = \{5, 3, [2]^2, 0\}$, Lemma 2.5 implies that B is the product of two graphs, say $B = B_1 \times B_2$. If $B_1 = K_1$, then we have $\mathrm{Spec}(B_2) = \{2, [1]^2, 0\}$. It follows that the number of spanning tree is $s(B_2) = \frac{2}{4}$, a contradiction. If $|VB_1|$=2, then $|VB_2| = 3$. Notice that the second largest Laplacian eigenvalue of B is 3, by Lemma 2.3, the maximum Laplacian eigenvalue of B_1 is 0. Thus, $B_1 = 2K_1$, and then $B_2 = 3K_1$. It is obvious that $B \times K_1 = G_1 \times G_2$.

Case 1.4.2. $n = 8$. Note that the maximum Laplacian eigenvalue of B is 6. Applying Lemmas 2.1, 2.6 and 2.7, we obtain

$$\alpha(B) \geq 6, \quad \Delta(B) \leq 4, \quad s(B) = 15.$$

If $t = 0$, then $s(B) = rs = 15$, i.e., $r = 3, s = 5$. Applying (5.3), we obtain $\alpha(B) < 6$, a contradiction. If $t = 1$, then $s(B) = rs = 15$, i.e., $r = 3, s = 5$. It is easily obtained that the degree sequence of B is

$$\deg(B) = (4, 3, [2]^5, 1),$$

then the degree sequence of $B \times K_1$ is

$$\deg(B \times K_1) = (8, 5, 4, [3]^5, 2),$$

whose square sum is 154. But the degree sequence of $G_1 \times G_2$ is

$$\deg(G_1 \times G_2) = ([7]^2, 5, [3]^3, [2]^3),$$

whose square sum is $162 \neq 154$, a contradiction to Lemma 2.2, since $B \times K_1$ and $G_1 \times G_2$ are L-cospectral. If $t = 2$, then $s(B) = rs - 1 = 15$ implies $r = 4, s = 4$, i.e., $\Theta_{4,2,4}$ is a subgraph of B. Note that $|VB| - |V\Theta_{4,2,4}| = 2$. Following the idea of the proof for Proposition 3.1, the square sum of the degree sequence of $B \times K_1$ is maximum iff $\deg(B) = ([4]^2, [2]^4, [1]^2)$. It follows that

$$\deg(B \times K_1) = (8, [5]^2, [3]^4, [2]^2),$$

whose square sum is 158. But the square sum of the degree sequence of $G_1 \times G_2$ is 162, a contradiction to Lemma 2.2. If $t \geq 3$, then (5.10) implies $t = 3$. It is impossible since $rs = s(B) + (t-1)^2 = 19$.

Case 1.4.3. $n = 10$. Note that the maximum Laplacian eigenvalue of B is 8. Applying Lemmas 2.1, 2.6 and 2.7, we obtain

$$\alpha(B) \geq 8, \quad \Delta(B) \leq 6, \quad s(B) = 16,$$

If $t = 0$, then $s(B) = rs = 16$, i.e., $r = 4, s = 4$. From (5.3), we obtain $\alpha(B) < 8$, a contradiction. If $t = 1$, then $s(B) = rs = 16$, and then $r = s = 4$. Thus $\Theta_{4,1,4}$ is a subgraph of B. Note that $|VB| - |V\Theta_{4,1,4}| = 3$. Then the square sum of the degree sequence of $B \times K_1$ is maximum iff the degree sequence of B is $(6, 3, [2]^5, [1]^3)$. It follows that

$$\deg(B \times K_1) = (10, 7, 4, [3]^5, [2]^3),$$

whose square sum is 222. Namely, the maximum square sum of the degree sequence of $B \times K_1$ is 222. But the square sum of the degree sequence of $G_1 \times G_2$ is 234, a contradiction to Lemma 2.2. If $t = 2$, then $s(B) = rs - 1 = 16$ has no solution. If $t \geq 3$, then (5.10) implies $t = 3$. Since $s(B) = rs - (t-1)^2 = 16$, we have $r = 4, s = 5$, i.e., $\Theta_{4,3,5}$ is a subgraph of B. Also consider the square sum of the degree sequence of $B \times K_1$. We can obtain that its maximum value is 226 only when

$$\deg(B \times K_1) = (10, 7, 5, [3]^4, [2]^4).$$

But the square sum of the degree sequence of $G_1 \times G_2$ is 234, a contradiction to Lemma 2.2.

Case 1.4.4. $n = 20$ or 40. We only disprove $n = 20$ in details. The argument to disprove $n = 40$ will be similar. Suppose $n = 20$. Note that the maximum Laplacian eigenvalue of B is 18. Applying Lemmas 2.1, 2.6 and 2.7, we obtain

$$\alpha(B) \geq 18, \quad \Delta(B) \leq 16, \quad s(B) = 18.$$

If $t = 0$ or 1, then $s(B) = rs = 18$, i.e., $r = 3, s = 6$. Applying (5.3) and (5.4), we obtain $\alpha(B) < 18$, a contradiction. If $t = 2$, then $s(B) = rs - 1 = 18$ has no solution. If $t \geq 3$, then we can also obtain (5.10), and then $t = 3$. Since $s(B) = rs - (t-1)^2 = 18$, there is no solution.

Case 1.5. $G_2 = C_3 + (n-4)K_1$. Clearly, $n \geq 4$. Since the Laplacian spectrum of C_3 is $\{[3]^2, 0\}$, we can obtain

$$\mathrm{Spec}(B) = \{n-2, [4]^2, [1]^{n-4}, 0\}.$$

It follows that the number of spanning trees of B is $s(B) = 16 - \frac{32}{n}$. Hence, $n = 4, 8, 16$, or 32.

Case 1.5.1. $n = 4$. It follows that $\mathrm{Spec}(B) = \{[4]^2, 2, 0\}$. Then we have $B = K_1 \times K_1 \times 2K_1$ by Lemma 2.5. It is easily seen that $B \times K_1 = G_1 \times G_2$.

Case 1.5.2. $n = 8$. Note that the maximum Laplacian eigenvalue of B is 6. Applying Lemmas 2.1, 2.6 and 2.7, we obtain

$$\alpha(B) \geq 6, \quad \Delta(B) \leq 4, \quad s(B) = 12.$$

If $t = 0$, then $s(B) = rs = 12$ implies that $r = 3, s = 4$, and then $\alpha(B) \leq 6$ by (5.3). Thus, $\alpha(B) = 6$, which forces the degree sequence of B is $\deg(B) = (4, 3, [2]^5, 1)$. Then the degree sequence of $B \times K_1$ is $\deg(B \times K_1) = (8, 5, 4, [3]^5, 2)$, whose square sum is 154. On the other hand, since $G_2 = C_3 + 4K_1$ and $G_1 = 2K_1$, then the degree sequence of $G_1 \times G_2$ is $\deg(G_1 \times G_2) = ([7]^2, [4]^3, [2]^4)$, whose square sum is 162. It is a contradiction by Lemma 2.2. If $t = 1$, then we also have $r = 3, s = 4$, i.e., $\Theta_{3,1,4}$ is a subgraph of B. Since $\Delta(B) \leq 4$, then the square sum of the degree sequence of $B \times K_1$ is maximum iff $\deg(B) = ([4]^2, [2]^4, [1]^2)$. It follows that $\deg(B \times K_1) = (8, [5]^2, [3]^4, [2]^2)$ whose square sum is $158 < 162$, a contradiction to Lemma 2.2. If $t = 2$, then $s(B) = rs - 1 = 12$ has no solution. If $t \geq 3$, then we can also obtain (5.10), and then $t = 3$. Since $s(B) = rs - (t-1)^2 = 12$, then $t = 3, r = s = 4$. Then B contains $\Theta_{3,2,4}$ as a subgraph. Since $\Delta(B) \leq 4$, then the square sum of the degree sequence of $B \times K_1$ is maximum iff $\deg(B) = ([4]^2, 3, [2]^2, [1]^3)$. It follows that $\deg(B \times K_1) = (8, [5]^2, 4, [3]^2, [2]^3)$, whose square sum is $160 < 162$, a contradiction.

Case 1.5.3. $n = 16$. Note that the maximum Laplacian eigenvalue of B is 14. Applying Lemmas 2.1, 2.6 and 2.7, we obtain

$$\alpha(B) \geq 14, \quad \Delta(B) \leq 12, \quad s(B) = 14.$$

If $t = 0$ or 1, then $s(B) = rs = 14$ has no solution. If $t = 2$, then $s(B) = rs - 1 = 14$ implies $r = 3, s = 5$, i.e., $\Theta_{3,2,5}$ is a subgraph of B. Since $|VB| - |V\Theta_{3,2,5}| = 0$ and and $\Delta(B) \leq 12$, then the square sum of $\deg(B \times K_1)$ is maximum iff $\deg(B) = (12, 4, [2]^4, [1]^{10})$. It follows that the square sum of $\deg(B \times K_1)$ is 526. But the square sum of $\deg(G_1 \times G_2)$ is 546. It is a contradiction to Lemma 2.2. If $t \geq 3$, then (5.10) implies $t = 3$. Since $s(B) = rs - (t-1)^2 = 14$, we have $r = 3, s = 6$, a contradiction to (5.1).

Case 1.5.4. $n = 32$. Note that the maximum Laplacian eigenvalue of B is 30. Applying Lemmas 2.1, 2.6 and 2.7, we obtain

$$\alpha(B) \geq 30, \quad \Delta(B) \leq 28, \quad s(B) = 15.$$

If $t = 0$, then $s(B) = rs = 15$, i.e., $r = 3, s = 5$. Applying (5.3), we obtain $\alpha(B) < 29$, a contradiction. If $t = 1$, then $s(B) = rs = 15$ implies $r = 3, s = 5$, i.e., $\Theta_{3,1,5}$ is a subgraph of B. Then, the square sum of the degree sequence of $B \times K_1$ is maximum iff $\deg(B) = (28, 3, [2]^5, [1]^{25})$. It follows that the degree sequence of $B \times K_1$ is $(32, 29, 5, [3]^4, [2]^{26})$, whose square sum is 2014. But the square sum of the degree sequence of $G_1 \times G_2$ is 2082, a contradiction to Lemma 2.2. If $t \geq 3$, then (5.10) implies $t = 3$. Then $s(B) = rs - (t-1)^2 = 15$ has no solution.

Case 2. $v_1 = 3$. Substituting $v_1 = 3$ into (5.7), we have $0 \leq e_1 + e_2 = -n + 7$, i.e., $n \leq 7$. From the assumption that $n \geq 2v_1 - 1$, we have $n \geq 5$. Then, $n = 5, 6$, or 7.

Case 2.1. $n = 5$. Substituting $v_1 = 3$ and $n = 5$ into (5.7), we have $e_1 + e_2 = 2$. Clearly, $v_2 = n + 1 - v_1 = 3$. It follows that

$$G = P_3 \times 3K_1, \quad \text{or} \quad G = (P_2 + K_1) \times (P_2 + K_1).$$

Since $G = P_3 \times 3K_1 = K_1 \times (2K_1 \times 3K_1)$, Lemma 2.4 implies $B = 2K_1 \times 3K_1$. Now consider $G = (P_2 + K_1) \times (P_2 + K_1)$. By Lemma 2.3, the Laplacian spectrum of G is $\{6, [5]^2, [3]^2, 0\}$. Then the Laplacian spectrum of B is $\{[4]^2, [2]^2, 0\}$, and then $s(B) = 64/5$, a contradiction.

Case 2.2. $n = 6$. Substituting $v_1 = 3$ and $n = 6$ into (5.7), we have $e_1 + e_2 = 1$. Clearly, $v_2 = 4$. It follows that

$$G = 3K_1 \times (P_2 + 2K_1), \quad \text{or} \quad G = 4K_1 \times (P_2 + K_1).$$

Case 2.2.1. $G = 3K_1 \times (P_2 + 2K_1)$. By Lemma 2.3, the Laplacian spectrum of B is $\{4, [3]^2, [2]^2, 0\}$. By Lemma 2.7, we have $\Delta(B) \leq 2$, a contradiction to that B is a bicyclic graph.

Case 2.2.2. $G = 4K_1 \times (P_2 + K_1)$. By Lemma 2.3, the Laplacian spectrum of B is $\{5, 3, [2]^3, 0\}$. Then Lemma 2.1 implies that $s(B) = 20$. Suppose $B \in \mathcal{B}(6, r, t, s)$. If $t = 0$, then $s(B) = rs = 20$, and then $r = 4, s = 5$, a contradiction to (5.2). If $t \geq 1$, then $\alpha(B) = rs - (t-1)^2 = 20$ and $r + s \leq n + t = t + 6$ has no solution, a contradiction.

Case 2.3. $n = 7$. Substituting $v_1 = 3$ and $n = 7$ into (5.7), we have $e_1 + e_2 = 0$. Clearly, $v_2 = 5$. Then $G = 3K_1 \times 5K_1$. By Lemma 2.3, the Laplacian spectrum of B is $\{[4]^2, [2]^4, 0\}$. Then $s(B) = 196/7$, a contradiction.

So far, we can conclude that, for all bicyclic graphs but $\Theta_{3,2,5}$ as in *Case 1.3.1*, we have $v_1 = 1$. The proof is complete by Lemma 2.4. ☐

From *Case 1.3.1* of Lemma 5.2 and Lemma 2.3, it is trivial to get the following result.

Corollary 5.3. *Graphs $\Theta_{3,2,5} \times K_m$ and $H_2 \times K_{m-1}$ are L-cospectral for all positive integers m, where H_2 is given in Figure* 2.

In the following, we will use induction to prove the last main result.

Theorem 5.4. *If a bicyclic graph B is L-DS and $B \neq \Theta_{3,2,5}$, then the product $B \times K_m$ is L-DS for all positive integer m.*

Proof. The idea to prove this theorem is similar as the proof of Theorem 3.3 or 4.4. We repeat some arguments of Theorem 3.3. The statement for $m = 1$ is given in Lemma 5.2. Now we assume $m \geq 2$. Let G be a graph L-cospectral to $B \times K_m = B \times \underbrace{K_1 \times K_1 \times \cdots \times K_1}_{m}$. By Lemma 2.5, G can be written as

$$G = G_0 \times G_1 \times \cdots \times G_m.$$

Fix notations as follows,

$$n = |VB|, \quad e_i = |EG_i|, \quad v_i = |VG_i| \quad \text{for} \quad i = 0, 1, \ldots, m. \tag{5.11}$$

Without loss of generality, assume $v_0 \geq v_1 \geq \cdots \geq v_m$. It is obvious that $\sum_{i=0}^{m} v_i = n + m$ by Lemma 2.2. In the following, we are going to prove $v_m = 1$ by contradiction. Now suppose $v_m \geq 2$. It follows that $v_i \geq 2$ for all $i = 0, \ldots, m$. Then we have $m + n = \sum_{i=0}^{m} v_i \geq 2(m+1)$, so $n \geq m + 2$. For convenience, we list those conclusions we have obtained,

$$m \geq 2, \quad v_0 \geq \cdots \geq v_m \geq 2, \quad m + n = \sum_{i=0}^{m} v_i, \quad n \geq m + 2. \tag{5.12}$$

Combining $v_0 \geq \cdots \geq v_m \geq 2$ with $\sum_{i=0}^m v_i = n + m$, by Proposition 3.1, we have

$$\sum_{i=0}^{m} v_i^2 \leq (n-m)^2 + 4m. \tag{5.13}$$

Since $\mathrm{Spec}(G) = \mathrm{Spec}(T \times K_m)$, Lemma 2.2 implies that G and $T \times K_m$ have the same number of edges. Counting the edges of both G and $T \times K_m$, we have

$$\sum_{i=0}^{m} e_i + \sum_{0 \leq i < j \leq m} v_i v_j = n + 1 + mn + \frac{m(m-1)}{2}. \tag{5.14}$$

Applying $\sum_{i=0}^m v_i = n + m$ to (5.14), we have

$$\sum_{i=0}^{m} e_i = \frac{1}{2}\left(\sum_{i=0}^{m} v_i^2 - n^2 - m\right) + n + 1. \tag{5.15}$$

Applying (5.13) to (5.15), it follows that

$$\sum_{i=0}^{m} e_i \leq (1-m)n + \frac{1}{2}(m^2 + 3m) + 1. \tag{5.16}$$

Note that, from (5.12), we have $1 - m < 0$ and $n \geq m + 2$. Then (5.16) implies

$$\sum_{i=0}^{m} e_i \leq -\frac{1}{2}(m^2 - m - 6). \tag{5.17}$$

Applying the fact $\sum_{i=0}^m e_i \geq 0$ to (5.17), it is easily obtained that $m = 2$ or 3.

Case 1. $m = 3$. Substituting $m = 3$ into (5.16) and (5.17), we have

$$0 = \sum_{i=0}^{m} e_i \leq -2n + 10.$$

It follows that $n \leq 5$. But $n \geq m + 2 = 5$ by (5.12). Hence, $n = 5$. It follows that $|VG| = \sum_{i=0}^3 v_i = 8$. Since $v_0 \geq \cdots \geq v_3 \geq 2$, then we have

$$v_0 = v_1 = v_2 = v_3 = 2, \text{ and } e_0 = e_1 = e_2 = e_3 = 0.$$

Thus, $G = 2K_1 \times 2K_1 \times 2K_1 \times 2K_1$. By Lemma 2.3, the minimal nonzero Laplacian eigenvalue of G is 2. But the minimal nonzero Laplacian eigenvalue of $B \times K_3$ is at least 3, a contradiction.

Case 2. $m = 2$. By (5.12), we have $n \geq m + 2 = 4$. On the other hand, (5.17) implies that

$$0 \leq \sum_{i=0}^{m} e_i \leq -n + 6.$$

Then we have $n = 4, 5$, or 6.

Case 2.1. $n = 4$. Then $|VG| = v_0 + v_1 + v_2 = 6$ and $v_0 \geq v_1 \geq v_2 \geq 2$. It follows that $v_0 = v_1 = v_2 = 2$. By (5.14), we have $e_0 + e_1 + e_2 = 1$. It follows that

$$G = K_2 \times K_2 \times 2K_1 = K_1 \times K_3 \times 2K_1.$$

By Lemma 2.4, we have $G = B \times K_3$.

Case 2.2. $n = 5$. Similar as above, we can obtain that

$$v_0 = 3, \quad v_1 = v_2 = 2, \text{ and } e_0 + e_1 + e_2 = 1.$$

Namely,

$$G = 3K_1 \times 2K_1 \times K_2, \text{ or } G = (P_2 + K_1) \times 2K_1 \times 2K_1.$$

If $G = 3K_1 \times 2K_1 \times K_2 = 3K_1 \times P_3 \times K_1$, applying Lemma 2.4, we have $G = B \times K_3$. If $G = (P_2 + K_1) \times 2K_1 \times 2K_1$, by Lemma 2.3, the Laplacian spectrum of G is $\{[7]^2, 6, [5]^2, 4, 0\}$, and then the Laplacian spectrum of B is $\{4, [3]^2, 2, 0\}$. So $s(B) = 72/5$, it is a contradiction.

Case 2.3. $n = 6$. We have $e_0 + e_1 + e_2 = 0$. Then

$$G = 4K_1 \times 2K_1 \times 2K_1, \text{ or } G = 3K_1 \times 3K_1 \times 2K_1.$$

The Laplacian spectrum of G is $\{[8]^2, [6]^2, [4]^3, 0\}$ or $\{[8]^2, 6, [5]^4, 0\}$. Then the Laplacian spectrum of B is $\{[4]^2, [2]^3, 0\}$ or $\{4, [3]^4, 0\}$. Namely, the maximum Laplacian eigenvalue of B is 4. By Lemma 2.7, $\Delta(B) \leq 2$, a contradiction.

Hence $v_m = 1$, i.e., $G_m = K_1$. Since $B \times K_m = B \times K_{m-1} \times K_1$, by Lemma 2.3, $B \times K_{m-1}$ is L-cospectral to $G_0 \times G_1 \times \cdots \times G_{m-1}$. Using the induction hypothesis on $m - 1$, $B \times K_{m-1}$ and $G_0 \times G_1 \times \cdots \times G_{m-1}$ are isomorphic. Hence, $B \times K_m = G$. This completes the proof. $\square$

From [14], we know all graphs $\Theta_{r,1,s}$ with $r, s \neq 3$ are determined by their Laplacian spectra, then Theorem 5.4 implies that $\Theta_{r,1,s} \times K_m$ with $r, s \neq 3$ are L-DS graphs.

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